CONTINUED FRACTIONS

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PREFACE

This paper has been written as a partial fulfillment of the requirements for reception of the Master of Science degree. The primary aim of the paper is to present a study of Continued Fractions and its Applications.

Chapters I and II stress the transformation of continued fractions, definition of a continued fraction and properties of convergents.

Chapter III contains the transformation of a Power series into a continued fraction using Euler identity and the expansion of Corresponding and Associated Continued Fractions with the Power series. Examples are used to illustrate the transformation.

Chapter IV contains the Convergence Theory of Continued Fractions, Convergence Test for Continued Fractions which are periodic in the Limit and Convergence Test for Continued Fractions with Arbitrary Coefficients.

And in the final chapter Continued Fractions Solution of Differential Equations.

The writer wishes to thank Dr. Lloyd K. Williams for his helpful suggestions and criticisms.

CHAPTER I

INTRODUCTION

1.0 Convergents

Basic Definition:

An expression of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}} \qquad (1.1)$$

is called a continued fraction. The fraction $\frac{a_n}{b_n}$ is called the n^{th} partial quotient of the continued fraction (1.1). A_n and b_n are the terms of the n^{th} partial quotient. $a_1, a_2, a_3, \ldots$ are called the partial numerators of the continued fraction, $b_1, b_2, b_3, \ldots$ are its partial denominators. All the coefficients of a continued fraction are finite. We shall assume that all partial denominators of a continued fraction are not equal to zero.

The terminating continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \equiv \frac{P_n}{Q_n}$$

is called the n^{th} convergent of the continued fraction. (1.1)

1.1 We derive the relations connecting the numerators and denominators of three consecutive convergents. From the definition of a continued fraction we have

$$\frac{P_0}{Q_0} = \frac{b_0}{1}, \quad \frac{P_1}{Q_1} = \frac{b_0 b_1 + a_1}{b_1},$$

$$\frac{P_2}{Q_2} = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2}} = b_0 + \frac{a_1 b_2}{b_1 b_2 + a_2} =$$

$$= \frac{b_0 b_1 b_2 + b_0 a_2 + a_1 b_2}{b_1 b_2 + a_2} = \frac{b_2 P_1 + a_2 P_0}{b_2 Q_1 + a_2 Q_0}.$$

Assume that

$$P_n = b_n P_{n-1} + a_n P_{n-2},$$
$$Q_n = b_n Q_{n-1} + a_n Q_{n-2}. \qquad (1.2)$$

Then $$\frac{P_n}{Q_n} = \frac{b_n P_{n-1} + a_n P_{n-2}}{b_n Q_{n-1} + a_n Q_{n-2}}$$

We prove that relations (1.2) are valid when n is replaced by n+1. For this we note that in order to progress from P_n/Q_n to P_{n+1}/Q_{n+1} one must replace b_n by b_n +

(a_{n+1}/b_{N+1}) Then

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{b_n P_{n-1} + \frac{a_{n+1}}{b_{n+1}} P_{n-1} + a_n P_{n-2}}{b_n Q_{n-1} + \frac{a_{n+1}}{b_{n+1}} Q_{n-1} + a_n Q_{n-2}}$$

$$= \frac{b_{n+1} P_n + a_{n+} P_{n-1}}{b_{n+1} Q_n + a_n Q_{n-1}}.$$

Consequently equation (1.2) is valid for all integer $n \geq 2$.

In order that relationships (1.2) should be valid for $n = 1$, we put, $P_{-1}=1$, $Q_{-1}==0$

1.3 We shall use the following scheme:

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} + \ldots,$$

$$\frac{1}{0} \quad \frac{b_0}{1} \quad \frac{P_1}{Q1} \qquad \frac{P_2}{Q_2} \qquad \frac{P_n}{Q_n} \quad \ldots.$$

to indicate the computation of the successive convergents.

CHAPTER II

TRANSFORMATION OF CONTINUED FRACTIONS

1. We multiply a_m, b_m and a_{m+1} by an arbitrary finite number $\rho_m (m=0, 1, \ldots, n, \ldots)$, differing from zero. It is clear that as a result of this, the value of the continued fraction is unaltered. Therefore the following identity is valid:

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} \ldots = \tag{1.4}$$
$$= b + \frac{\rho_1 A_1}{\rho_1 b_1} + \frac{\rho_1 \rho_2 A_2}{\rho_2 b_2} + \ldots + \frac{\rho_{n-1}\rho_n A_n}{\rho_n b_n} + \ldots$$

With this P_1 and Q_1 are replaced by $\rho_1 P_1$ and $\rho_1 Q_1$ respectively; P_2 and Q_2 – by $\rho_1 \rho_2 P_2$ and $\rho_1 \rho_2 Q_2, \ldots,$ P_n and Q_n $\rho_1 \rho_2 \ldots \rho_n P_n$ and $\rho_1 \rho_2 \ldots \rho_n Q_n$.

By means of this transformation all partial denominators $b_1, b_2, \ldots$ of the continued fraction (1.1) can always be made positive.

2. The operation of contraction and extension:

Special cases of the continued Fraction (1.1):

Put, in particular,

$$K_n = \frac{P_{2n}}{Q_{2n}}$$

From (1.2) we have:

$$P_{2n} = b_{2n}P_{2n-1} + A_{2n}P_{2n-2},$$

$$P_{2n-1} = b_{2n-1}P_{2n-2} + A_{2n-1}P_{2n-3},$$

$$P_{2n-2} = b_{2n-2}P_{2n-3} + A_{2n-2}P_{2n-4}.$$

Multiplying these equations by b_{2n-2}, $b_{2n}b_{2n-2}$ and $-A_{2n-1}b_{2n}$ respectively and adding the derived products, we have:

$$b_{2n-2}P_{2n} = (a_{2n}b_{2n-2} + b_{2n}b_{2n-1}b_{2n-2} + A_{2n-1}B_{2n})\ P_{2n-2} -$$
$$-A_{2n-1}A_{2n-2}b_{2n}P_{2n-4} \quad (n=2,3,\dots)$$

Similar relationships connect $Q_{2n}, Q_{2n-2}, Q_{2n-4}$.

The derived relationships connect the numerators and denominators of three successive convergents of the contrasted continued fractions. Therefore, (c.f.(1.2.) the coefficient of P_{2n-2} divided by b_{2n-2} is the n^{th} partial denominator of the contracted continued fraction, and the coefficient of P_{2n-4} divided by b_{2n-2} is the n^{th} partial numerator of the contracted continued fraction.

Moreover $\quad \dfrac{P_2}{Q_2} = b_0 + \dfrac{a_1b_1}{b_1b_2 + a_2}$

From this the partial numerators of the continued fraction become

$$a_1b_2, \ -\frac{a_2a_3b_4}{b_2}, \ -\frac{a_4\ a_5\ b_6}{b_4}, \ \dots,$$

$$- \frac{a_{2n-2}a_{2n-1}b_{2n}}{b_{2n-2}}, \ldots,$$

and its partial denominators become

$$b_1b_2+a_2, \quad \frac{(b_2b_3+a_3)b_4+b_2a_4}{b_2}, \quad \frac{(b_4\,b_5+a_5)b_6\pm b_4a_6}{b_4},$$

$$\ldots, \frac{(b_{2n-2}b_{2n-1}+a_{2n-1})\,b_{2n}+b_{2n-2}a_{2n}}{b_{2n-2}}, \ldots$$

Applying transformation (1.4) to the contracted expansion we obtain:

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} + \ldots =$$

$$= b_0 + \frac{a_1\,b_2}{b_1\,b_2+a_2} - \frac{a_2\,a_3\,b_4}{(b_2\,b_3+a_3)b_4+b_2a_4 -}$$

$$- \frac{a_4\,a_5\,b_2\,b_6}{(b_4\,b_5+a_5)\,b_6+b_4a_6} - \ldots$$

$$\ldots - \frac{a_{2n-2}a_{2n-1}\,b_{2n-4}\,b_{2n}}{(b_{2n-2}b_{2n-1}+a_{2n-1})\,b_{2n}+b_{2n-2}\,a_{2n}} - \ldots \qquad (1.10)$$

5. We now consider further identity transformation of the continued fraction

(1.1) We write the identity

$$K = b + \frac{a_1}{b_1} + \frac{a_2}{b\,2} + \ldots = \frac{b_0}{1} + \frac{c_1}{d_1} + \frac{c_2}{d_2}$$

$$+ \ldots \qquad (1.11)$$

and discuss the relationships which exist between the coefficients of these continued fractions. For this we denote Kby $b_o/(1=K,)$ Then

$$K = b_0 + \frac{b_0}{1 + k} - b_0 = b_0 - \frac{b_{oK1}}{1 + k} =$$

$$= b_0 - \frac{b_0}{\frac{1}{K} + 1} = b_0 - \frac{b_0 c_1 d_2}{\frac{c_1 d_2}{K_1} + c_1 d_2}$$

But according to (1.10)

$$K_1 = \frac{c_1 d_2}{d_1 d_2 + c_2} - \frac{c_2 c_3 d_4}{(d_2 d_3 + c_3) d_4 + d_2 c_4} -$$

$$- \frac{c_4 c_5 d_2 d_6}{(d_4 d_5 + c_5) d_6 d_4 c_6 - \dots}$$

$$\dots - \frac{c_{2m-2} c_{2n-1} d_{2n-4} d_{2n}}{(d_{2n-2} d_{2n-1} + c_{2n-1}) d_{2n} d_{2n-2} c2n - \dots}$$

Consequently

$$K = b_0 - \frac{b_0 c_1 d_2}{(d_1 + c_1) d_2 + c_2} - \frac{c_2 c_3 d_4}{(d_2 d_3 + c_3) d_4 + d_2 c_4}$$

$$- \frac{c_4 c_5 d_2 d_6}{(d_4 d_5 + c_5) d_6 + d_4 c_6} - \dots$$

$$\dots - \frac{c_{2n-2} c_{2n-1} d_{2n-4} d_{2n}}{(d_{2n-2} d_{2n-1} + c_{2n-1}) d_{2n} + d_{2n-2} c_{2n}} \qquad (1,12)$$

But on the other hand, K satisfies (1.10). From this, assuming that as a result of transformation (1.4) corresponding coefficients of continued fractions (1.10) and

(1.12) are made equal. We have:

$$-b_0\, c_1\, d_2 = a_1\, b_2,$$

$$c_2\, c_3\, d_4 = a_2\, a_3\, b_4, \qquad (1,13)$$

$$c_{2n-2}c_{2n-1}d_{2n-4}d_{2n} = a_{2n-2}a_{2n-1}b_{2n-4}b_{2n} \qquad (n=2,3\ldots)$$

and $c_1\, d_2 + d_1\, d_2 + c_2 = b_1\, b_2 + a_2$,

$$(d_{2n-2}d_{2n-1}+c_{2n-1})\, d_{2n}+d_{2n-2}c_{2n} = (1.14) \qquad (1.14)$$

$$= (b_{2n-2}b_{2n-1}+ a_{2n-1})\, b_{2n}+b_{2n-2}a_{2n} \qquad (n=2,3,\ldots)$$

We apply (1.11) to a special case of the continued fraction (1.1).

Let $b_0=1 \quad c_1= -a_1 \quad d_n = b_n,$

$$c_{2n\pm1}=a_{2n}, \quad c_{2n}=a_{2n+1} \qquad (n=1,2,\ldots)$$

Then equations (1.13) evolve into identities and equations (1.14) become

$$-a_1\, b_2 + a_3 = a_2$$

$$a_{2n-2}b_{2n}+a_{2n+1}b_{2n-2}=a_{2n-1}b_{2n}+a_{2n}b_{2n-2} \qquad (n=2,3,\ldots)$$

From this

$$a_1 = \frac{a_3 - a_2}{b_2}, \quad \frac{a_{2n-1}-a_{2n-2}}{b_{2n-2}} = \frac{a_{2n+1}- a_{2n}}{b_{2n}}$$

$(n=2,3,\ldots)$

Hence with the conditionx

$$a_1 = \frac{a_3 - a_2}{b_1} = \frac{a_5 - a_4}{b_4} = \ldots \frac{a_{2n+1}-a_{2n}}{b_{1n}}$$

the identity

$$1 + \frac{a_1}{b_1} + \frac{a_2}{b_1} + \ldots + \frac{a_m}{b_{n+} \ldots} =$$

$$= \frac{1}{1} - \frac{a_1}{b_1} + \frac{a_3}{b_1} + \frac{a_2}{b_3} + \ldots + \frac{a_{2n+1}}{b_{2n}} + \frac{a_{2n}}{b_{2n+1}} + \ldots \quad (1.16)$$

obtains.

6. We consider a further special case of transformation (1.11). Let

$$b_0 = 1, \quad c_n = -a_n, \quad d_{2n} = b_{2n}, \quad d_{2n-1} = \lambda_n b_{2n-1}$$

$$(n=1,2,\ldots,\ \lambda_n \neq 1)$$

These equations (1.13) degenerate into identities and equations (1.14) become

$$-a_1 b_2 + \lambda_1 b_1 b_2 = a_2 = b_1 2_2 + a_2 ,$$

$$(\lambda_n b_{2n-2} b_{2n-1} = a_{2n-1} +) b_{2n} - b_{2n-2} a_{2n} = \qquad (n=2,3,\ldots)$$

$$= b_{2n-2} b_{2n-1} + a_{2n-1} b_{2n} + b_{2n-2} a_{2n}$$

From this

$$\lambda_1 b_1 = \frac{a_1 b_2 + b_1 b_2 + 2 a_2}{b_2}$$

$$\lambda_n b_{2n-1} = \frac{b_{2n-2} b_{2n-1} b_{2n} + 2 a_{2n-1} b_{2n} + 2 a_{2n} b_{2n-2}}{B_{2m-2} B_{2m}}$$

in this way, equations (1.11) becomes

$$1 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots = \frac{1}{1} - \frac{\dfrac{a_1}{a_1 b_2 + b_1 b_2 + 2a_2} \quad \dfrac{a_2}{b_2}}{b_2} -$$

$$\frac{\dfrac{a_3}{b_2 b_3 b_4 + 2a_3 b_4 + 2a_4 b_2}}{b_2 b_4} - \frac{a_4}{b_4} - \frac{a_5}{b_4 b_5 b_6 + 2a_5 b_6 + 2a_6 b_4}$$

$$- \frac{a_6}{b_{2n} - \dots}$$

$$1 \quad \dots - \frac{a_{2n-1}}{b_{2n-2} b_{2n-1} b_{2n} + 2a_{2n-1} b_{2n} + 2a_6 b_4} - \frac{a_{2n}}{b_6}$$

$$\dots - \frac{\dfrac{a_{2n-1}}{b_{2n-2} b_{2n-1} b_{2n} + 2a_{2n-1} b_{2n} + 2a_{2n} b_{2n-2}}}{b_{2n-2} b_{2n}} - \frac{a_{2n}}{b_{2n}} - \dots$$

i.e.

$$1 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots =$$

$$\frac{1}{1} + \frac{a_1 b_2}{a_1 b_2 + b_1 b_2 \pm 2a_2} - \frac{a_2}{1} - \frac{a_3 b_4}{b_2 b_3 b_4 + 2a_3 b_4 \pm 2a_4 b_2}$$

$$- \frac{a_4 b_2}{1} - \frac{a_5 b_2}{b_4 b_5 b_6 + 2a_5 b_6 + 2a_6 b_4} - \frac{a_6 b_4}{1}$$

$$- \dots \qquad (1.17)$$

With the help of transformation (1.4) we reduce the continued fraction (1.1) to a form which all partial denominators are equal to 1. For this we put $P_n = 1/b_n$ $(n=1,2,\ldots)$. The continued fraction (1.1) becomes for $b_0 = 0$

$$\frac{c_1}{1} + \frac{c_2}{1} + \ldots + \frac{c_n}{1} + \ldots \tag{1.18}$$

where $c_1 = \dfrac{a_1}{b_1}$, $c_n = \dfrac{a_n}{b_{n-1}b_n}$ $(n=2,3,\ldots)$

In this P_n and Q_n are replaced by

$$\frac{P_n}{b_1 b_2 \ldots b_n} \quad \text{and} \quad \frac{Q_n}{b_1 b_2 \ldots b_n}$$ respectively.

CHAPTER III

THE TRANSFORMATION OF SERIES INTO CONTINUED FRACTIONS

We shall distinguish the forms which a given power or numerical series may be transformed into a continued fraction. As an example of the transformation of power series into an equivalent continued fraction we may quote the following identity of Euler.[1]

$$c_0 + c_1x + c_2x^2 + \dots + c_nx^n + \dots =$$

$$= c_0 + \frac{c_1x}{1}\;\frac{\frac{c_2}{c_1}x}{-1+\frac{c_2}{c_1}x} - \frac{\frac{c_3}{c_2}x}{1+\frac{c_3}{c_2}x} - \dots - \frac{\frac{c_n}{c_{n-1}}}{1+\frac{c_n}{c_{n-1}}x} - \dots$$

One can write this formula as:

$$\sum_{n=0} c_nx^n = \frac{c_0}{1} - \frac{\frac{c_1}{c_0}x}{1 \pm \frac{c_1}{c_0}x}\;\frac{\frac{c_2}{c_1}x}{-1+\frac{c_2}{c_1}x}\;\frac{\frac{c_3}{c_2}x}{-1+\frac{c_3}{c_2}x} - \dots \frac{\frac{c_n}{c_{n-1}}x}{-1+\frac{c_n}{c_{n-1}}x} - \dots$$

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[1] Khovanskii, A.K., The Application of Continued Fractions and Their Generalizations to Problems in Approximation Theory, (The Netherlands: P. Noordhoff, Ltd. Groningen, 1963), pp. 23-24.

Applying transformation (1.4) to the right hand side of this second identity, we obtain the formula

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + \frac{c_1 x}{1} - \frac{c_2 x}{-c_1 + c_2 x} - \frac{c_1 c_3 x}{-c_2 + c_3 x} - \cdots$$

$$\cdots - \frac{c_{m-2} c_n x}{c_{n-1} + Cnx} - \cdots = \frac{c_0}{1} - \frac{c_1 x}{c_2 + c_3 x} - \cdots$$

$$- \frac{c_0 c_2 x}{c_1 + c_2 x} - \frac{c_1 c_3 x}{c_2 + c_3 x} - \cdots - \frac{c_{n-2} c_n x}{c_{n-1} + c_n x} - \cdots$$

Since this series has been transferred into an equivalent continued fraction, then the n^{th} convergent of the continued fraction standing on the right hand side of Identity is identically equal to the sum of the first n+1 terms of the series standing on the left hand side. (2.1)

Example: (1)

$$\text{Arctan} = \frac{x}{} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots =$$

$$= \frac{x}{1 + \dfrac{\frac{x^2}{3}}{1 - \frac{x^2}{3} + 1}} + \frac{\frac{3}{5}x^2}{-\frac{3}{5}x^2} + \cdots + \frac{\frac{2n-1}{2n+1}x^2}{1 - \frac{2n-1}{2n+1}x^2 + \cdots}$$

i.e

$$\text{Arctan}_x = \frac{x}{1} \; \frac{x^2}{+3 - x^2} + \frac{9x^2}{5 - 3x^2 + \cdots}$$

$$\ldots + \frac{(an-1)^2x^2}{2n+1-(2n-1)x^2} + \ldots \qquad (2.2)$$ [1]

From this, when x=1, we have:

$$\frac{\pi}{4} = \frac{1}{1} + \frac{1}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \ldots + \frac{(2n-1)^2}{2} + \ldots \qquad (2.3)$$

$$\frac{0}{1} \quad \frac{1}{1} \quad \frac{2}{3} \quad \frac{13}{15} \quad \frac{76}{105} \quad \frac{789}{945}$$

(1.3)

Applying identity (1.17) to (2.2) we obtain:

$$\frac{\text{Arctan } x}{x} = \frac{1}{1} \quad \frac{x^2}{3-x^2} + \frac{9x^2}{5-3x^2} + \ldots =$$

$$= 1 - \frac{x^2(5-3x^2)}{(5-3x^2)x + (3-x^2)(5-3x^2) + 18x^2} \quad \frac{9x^2}{1}$$

$$- \ldots \quad = 1 - \frac{5x^2-3x^4}{15+9x^2} - \frac{9x^2}{1} - \ldots$$

i.e.

$$\text{Arctan } x = x - \frac{5x^3-3x^5}{15+9x^2} - \frac{9x^2}{1} - \ldots$$

$$\frac{x}{1} \quad \frac{15x + 14x^3 + 3x^5}{15+9x} \quad \frac{15x - 5x^3 + 3x^5}{15}$$

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[1] *Ibid.*, p.24.
[2] *Ibid.*, p. 25.
[3] *Ibid.*, p. 26.

In this way we have, as a result of the transformation of a power series into an equivalent continued fraction, derived rational approximation for Arctan X. In view of the complexity we shall not set out the general term of the expansion derived. But for expansion (2.3) we have:

$$\frac{4}{\pi} = 1 + \frac{1}{2} + \frac{3^2}{2} + \ldots + \frac{(2n-1)^2}{2} + \ldots =$$

$$= \frac{1}{1} - \frac{2.1}{2+4+13 -} \;\frac{9}{1} - \frac{2.25}{8 + 4.25 + 4.49 -}$$

$$- \frac{2.49}{1} \;\frac{2.81}{-8+4.81 \pm 4.121} - \ldots - \frac{2(4n-1)^2}{1} -$$

$$\frac{-2(4n+1)^2}{-8+4(4n+1)^2+4(4n+3)^2 - \ldots}$$

From this

$$\frac{\pi}{4} = 1 - \frac{2}{24 -} \;\frac{9}{1} \;\frac{25}{-152} \;\frac{49}{-1} \;\frac{81}{-408 - \ldots}$$

$$\frac{1}{1} \;\frac{22}{24} \;\frac{13}{15} \;\frac{1426}{1680} \;\frac{729}{945}$$

$$\ldots \frac{(4n-1)^2}{1} - \frac{(4n+1)^2}{8(2n+8n+3) - \ldots}$$ [1]

From the power series

$$A_0 + A_1x + A_2x^2 + \ldots$$

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[1] *Ibid.*, p.26.

one can find a continued fraction such that the expansion of its n^{th} convergent in a power series will coincide with the Original power series as far as the term containing x^n. Such a continued fraction is spoken of as corresponding to the given series. Customarily it is expressed in one of the following forms:

$$b_0 + \frac{a_1 x}{b_1} + \frac{a_2 x}{b_2} + \ldots; \qquad \frac{b_0}{1} + \frac{c_1 x}{d_1} + \frac{c_2 x}{d_2} + \ldots$$

Contracting the corresponding continued fractions, we obtained a continued fraction, the expansion of the n^{th} convergent of which in a power series coincide with the Original series as far as the term in x^{2n}. Such continued fractions are said to be associated with the given series.

Great importance attaches to the fact that the convergence behavior of a power or numerical series and that of its corresponding continued fraction is quite different. Both may converge, both diverge, or one may converge whilst the other diverges.

It is important to remark that one can even transform a power series with a convergence radius equal to zero into corresponding continued fraction which converges in a certain domain. One can express the coefficients of the corresponding fractions in terms of the coefficients in the terms of the Original Power series, but these expressions involve determinants of high order. We shall use a more convenient notation for this method of calculation:

Let

$$f(x) = \frac{\alpha_{10} + \alpha_{11}x + \alpha_{12}x^2 + \alpha_{13}x^3 + \ldots}{\alpha_{00} + \alpha_{02}x + \alpha_{02}x^2 + \alpha_{03}x^3 + \ldots}$$

Then

$$f(x) = \cfrac{1}{\cfrac{\alpha_{00}}{\alpha_{10}} + \cfrac{\alpha_{00} + \alpha_{01}x + \alpha_{02}x^2 + \ldots}{\alpha_{10} + \alpha_{11}x + \alpha_{12}x^2 + \ldots} - \cfrac{\alpha_{00}}{\alpha_{10}}}$$

$$= \cfrac{\alpha_{10}}{\alpha_{00} + x\,\cfrac{\alpha_{20} + \alpha_{21}x + \alpha_{22}x^2 + \ldots}{\alpha_{10} + \alpha_{11}x + \alpha_{12}x^2 + \ldots}}$$

$$= \cfrac{\alpha_{10}}{\alpha_{00} + x\,\cfrac{\alpha_{20}}{\alpha_{10} + x\,\cfrac{(\alpha_{20}\alpha_{11} - \alpha_{10}\alpha_{21}) + (\alpha_{20}\alpha_{12} - \alpha_{10}\alpha_{22})x + \ldots}{\alpha_{20} + \alpha_{21}x + \alpha_{22}x^2 + \ldots}}}$$

$$= \frac{\alpha_{10}}{\alpha_{00}} + \frac{\alpha_{20}x}{\alpha_{10}} + \frac{\alpha_{30}x}{\alpha_{20}} + \ldots$$

The computations are conveniently set out in the following scheme:

$$\begin{array}{cccc} \alpha_{00} & \alpha_{01} & \alpha_{02} & \ldots \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \ldots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \ldots \\ \alpha_{30} & \alpha_{31} & \alpha_{32} & \ldots \end{array}$$

Here $\alpha_{mn} = \alpha_{m-1,0}\,\alpha_{m-2,\,n+1} - \alpha_{m-2,0}\,\alpha_{m-1,\,n+1}$

Example:

(1) The Continued fraction expansion of the expression

$$\frac{1 - x}{1 - 5x + 6x^2} \qquad (x < 1/3)$$

we have:

$$\begin{array}{lll} 1 & -5 & 6 \\ 1 & -1 & \\ -4 & 6 & \\ -2 & & \\ -12 & & \end{array}$$

$$\frac{1 - x}{1 - 5x \pm 6x^2} = \frac{1}{1} - \frac{4x}{1} - \frac{2x}{4} - \frac{12x}{2} =$$

$$= \frac{1}{1} - \frac{4x}{1} + \frac{x}{2} - \frac{3x}{1}$$

$$\frac{0}{1} \quad \frac{1}{1} \quad \frac{1}{1-4x} \quad \frac{2 + x}{2 - 7x}, \quad \frac{2 - 2x}{2 - 10x + 12x^2} \tag{1}$$

(2) Consider the continued fraction expansion of the expression

$$\frac{1}{1 - 3x + 3x^2 - x^3} \qquad (x < 1)$$

we have:

$$\begin{array}{llll} 1 & -3 & 3 & -1 \\ 1 & & & \\ -3 & 3 & -1 & \\ -3 & 1 & & \end{array}$$

-6 3

3

9

$$\frac{1}{(1-x)^3} = \frac{1}{1} - \frac{3x}{1} - \frac{3x}{-3} - \frac{6x}{-3} + \frac{3x}{-6} + \frac{9x}{3}$$

$$= \frac{1}{1-} \frac{3x}{1} + \frac{x}{1} - \frac{2x}{3} + \frac{x}{2} - \frac{x}{1}$$

$$\frac{0}{1} \quad \frac{1}{1} \quad \frac{1}{1-3x} \quad \frac{1+x}{1-2x} \quad \frac{3+x}{3-8x+6x} \quad \frac{6+3x+x^2}{6-15x+10x^2} \quad \frac{6}{6-18x+18x^2-6x^3}$$

Let $2,0 = 0$ Then

$$f(x) = \cfrac{\alpha_{10}}{\alpha_{00} + x^2 \cfrac{\alpha_{21} + \alpha_{22}x + \dots}{\alpha_{10} + \alpha_{11}x + \alpha_{12}x^2 + \dots}}$$

The fraction $\dfrac{\alpha_{21} + \alpha_{22}x + \dots}{\alpha_{10} + \alpha_{11}x + \alpha_{12}x^2 + \dots}$

is expanded as in section b, and thus we come to the identity,

$$f(x) = \frac{\alpha_{10}}{\alpha_{20}} + \frac{\alpha_{21}x^2}{\alpha_{10}} + \frac{\alpha'_{31}x^2}{\alpha'_{21}} + \frac{\alpha'_{41}x}{\alpha'_{31}} + \dots$$

The computations are set out in the following schemes:

$$\begin{array}{llll}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \dots \\
\alpha_{10} & \alpha_{11} & \alpha_{12} & \dots \\
0 & \alpha_{21} & \alpha_{22} & \dots \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \dots. \\
\alpha'_{31} & \alpha'_{32} & \alpha'_{33} & \dots. \\
\alpha'_{41} & \alpha'_{42} & \alpha'_{43} & \dots
\end{array}$$

Here
$$\alpha'_{31} = \alpha_{21}\alpha_{11} - \alpha_{10}\alpha_{22},$$
$$\alpha'_{32} = \alpha_{21}\alpha_{12} - \alpha_{10}\alpha_{23},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha'_{41} = \alpha'_{31}\alpha_{22} - \alpha'_{32}\alpha_{21},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus, if $\alpha_{20} = o$, then the fourth row of the scheme is derived by means of a displacement of the third row by one place to the left, the fifth row is derived from the combination of the fourth and second rows by means of the general rule, the sixth row- by a combination of the fifth and fourth, and so on.

Exactly in the same way, if $\alpha_{k,0} = $) then the $(K+2)^{th}$ row of the scheme is derived by means of a displacement of the $(k+1)^{th}$ row by one place to the left; $(K+3)^{rd}$ row is derived by a combination of the $(K+2)^{nd}$ and K^{th} by means of the general rule, the $(K+4)^{th}$ row from a combination of the $(K+3)^{rd}$ and $(K+2)^{th}$, and so on. The expansion in this case becomes

$$f(x) = \frac{\alpha_{10}}{\alpha_{00}} + \frac{\alpha_{20}x}{\alpha_{10}} + \frac{\alpha_{30}x}{\alpha_{20}} + \cdots$$
$$\cdots + \frac{\alpha_{k-1,0}x}{\alpha_{k-2,0}} + \frac{\alpha_{k-1}x^2}{\alpha_{k-1,0}} + \frac{\alpha_{k+1,1}x}{\alpha_{k,1}} + \frac{\alpha_{K+2,1}x}{\alpha_{K+1,1}} + \cdots$$

(3) Example: Consider the continued fraction expansion of

the expression $\dfrac{1 - 3x^2}{1 - x^2 - 4x^4}$ $\left(x^2 < \dfrac{17 - 1}{8}\right)$

We have:

$$\begin{array}{ccccc} 1 & 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & -3 & \\ 0 & -1 & 3 & -4 & \\ -1 & 3 & -4 & & \\ -3 & 3 & -4 & & \\ -5 & 15 & 3 & & \\ 25 & -15 & & & \end{array}$$

300
$\underline{-}300.15$

$$\frac{1 - 3x^2}{1 - x^2 - 4x^4} = \frac{1}{1-}\ \frac{x^2}{1}\ \frac{3x}{-1-}\ \frac{5x}{-3+}\ \frac{25x}{-5}+\frac{300x}{25}$$

$$= \frac{15x}{1} = \frac{1}{1} - \frac{x^2}{1} + \frac{3x}{1} - \frac{5x}{3} +$$

$$\frac{0}{1}\quad \frac{1}{1}\quad \frac{1}{1-x^2}\quad \frac{1 + 0x}{1 + 3x - x^2}\quad \frac{3 + 4x}{3 + 4x - 3x^2 + 5x^3}$$

$$+\frac{5x}{1} = \frac{12x}{5} - \frac{3x}{1}$$

$$\frac{3 + 9x + 15x^2}{3 + 9x + 12x^2}\qquad \frac{15 + 9x + 27x^2}{15 + 9x \pm 12x^2 + 36x^2 - 60x^4}$$

$$\frac{15 - 45x^3}{15 - 15x^2 - 60x^4}\quad (1)$$

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[1] *Ibid.*, p. 31.

CHAPTER IV

CONVERGENCE THEORY OF CONTINUED FRACTIONS

1. Definition: The continued fractions for which $\lim_{n \to \infty}$ exists and is finite are called convergent. The value of the continued fraction is then equal to this limit.

2. If $\lim_{n \to \infty} \frac{P_n}{Q_n} = \infty$ or $\lim \frac{P_n}{Q_n} = -\infty$, then the continued fraction is called inessentially divergent. If $\lim_{n \to \infty} \frac{P_n}{Q_n}$ does not exist then the continued fraction is called essentially divergent.

The continued fraction $\left[\frac{a_v}{b_v}\right]_1^{\infty}$, is called <u>absolutely convergent</u>, if for all $m \geq 1$ the continued fractions $\left[\frac{a_v}{b_v}\right]_m^{\infty}$ converge. However, if for any value of m the latter continued fraction diverges, then the continued fraction $\left[\frac{a_v}{b_v}\right]_1^{\infty}$ is called <u>Conditionally Convergent</u>.

The concept of the <u>Uniform Convergent</u> of a continued fraction is extremely important.

Definition: If the coefficients of a continued fraction are functions of a finite or infinite number of variables, then this continued fraction is called uniformly convergent over a set E of variation of these variables if its convergents $\frac{P_n}{Q_n}$ are in E and converge uniformly to a limit, i.e. if for any $\epsilon > 0$, one can find a number N such that for $n \geq N$, $Q_N \neq 0$ and the inequality

$$\left| \frac{P_n}{q_n} - \lim_{\lambda \to \infty} \frac{P_\lambda}{q_\lambda} \right| < \epsilon \qquad (2.1)$$

is satisfied over all the set E.

Theorem I: The Uniform Convergence of the Continued Fraction (2.2)

$$\frac{c_1}{1} + \frac{c_2 x}{1} + \dots \frac{c_n x}{1} + \dots \quad (c_n \quad 0;$$

n=2, 1, ...)

over the set E is sufficient for the convergence over the set E of the continued fraction to the function K(x) of which it is the continued fraction expansion.

Theorem II: If the continued fraction $Km(x) = \left[\frac{c_\nu x}{1} \right]_{m+1}^{\infty}$ be uniformly convergent in a domain T containing the Origin as an interior point. According to theorem 1, Km(x) converges in T to a regular single valued Analytic function.

CONVERGENCE TEST FOR CONTINUED FRACTIONS WITH ARBITRARY COEFFICIENTS

1. The convergence of the series $\sum_{n=1}^{\infty} r_1 r_2 \dots r_n$ is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} (p_2-1)\,(p_3-1)\;.\;.\;.\;(p_{n+1}-1).$$

From this we have the following Theorem:

The set of Conditions

1. $p_1=1$

2. $|c_n| \leq \dfrac{p_{n-1}}{p_{n-1}\,p_n}$ $\qquad (n=2,3,\dots)$

3. The series $\sum_{n=1}^{\infty} (p_2-1)\,(p_3-1)\,\dots\,(p_{n+1}-1)$

converges is sufficient for the uniform convergence of the Continued Fraction (1.18) and for the inequality

$$|K| \leq 1+\sum_{n=1}^{\infty} (p_2-1)\dots(p_{n+2}-1)$$

to be valid.

From the condition

$$|c_n| \leq \frac{n^2-(k-1)^2}{4n^2-1} \qquad (n=2,3,\dots) \qquad (3.1)$$

is sufficient for the uniform convergence of the continued fraction (1.18)

When $R = \frac{1}{2}$, we have $P_n = 2$. In this case condition (3.1) becomes

$$\left|c_n\right| \leq \tfrac{1}{4} \quad (n = 2,3,\ldots) \qquad (3.2)$$

From (3.2) follows the Theorem:

The Condition

$$0 \leq c_n \leq g \quad (n=2,3,\ldots) \qquad (3.3)$$

is sufficient for the continued fraction $K = \left[\frac{1}{1}, \frac{c_n z}{1}\right]_2^{\infty}$

to converge in the circle $|z| < \frac{1}{4g}$ to a regular analytic non-rational function, and for the series corresponding to this continued fraction to do the same, and furthere more for $\left|K - \frac{4}{3}\right| \leq 2/3$.

CONVERGENCE TESTS FOR CONTINUED FRACTIONS WHICH ARE PERIODIC IN THE LIMIT

Definition: Continued fraction of the form $\left[\frac{a_v}{b_v}\right]_1^{\infty}$

for which $a_v \neq 0$, $\lim_{V\to\infty} a_v = a$, $\lim_{V\to\infty} b_v = b$, are called periodic in the limit.

Definition: If C is an upper bound for a set S and if any other upper bound $C \leq d$. Then C is the least upper bound or Supremun (sup) of S.

Definition: The interior of the circle is defined by

$$|z - z_0| < r$$

Definition: An E- neighborhood of z_0 is the set of complex numbers z such that $|z - z_0| < \epsilon$

Definition: A domain is an open, arcwise connected set.

Definition: A set S is arcwise connected if any pair of points z_1, and z_2 of S may be joined by a polygonal line, all the points of which are in S.

Theorem: 1. The condition $\limsup_{V\to\infty} |c_v| \leq g$ is sufficient for the continued fraction

$$\left[\frac{c_1}{1},\ \frac{c_v}{1}\right]_{2}^{\infty} \qquad (4.1)$$

to be convergent in the circle $|z| < \frac{1}{4}g$ (excepting possibly at certain poles) to a regular Analytic function, the pples of the latter being points of inessential divergences of the continued fraction(4.1). In the neighborhood of the Origin of this function is equal to the series corresponding to the continued fraction (4.1).

Theorem 2. The condition $\lim_{v \to \infty} c_v = 0$ is sufficient for the continued fraction (4.1) to converge uniformly in any finite domain with the exception of a finite number of points of inessential divergence, to an analytic function which is regular in the neighborhood of the Origin, and in the remaining part of the domain is regular, with the exception of the above mentioned points of inessential divergence of the continued fraction (4.1) which are poles of the function. The point $z = \infty$ is an essential singularity of this function.

Theorem 3: The condition $\lim_{v \to \infty} C_v = C \neq 0$ is sufficient for the uniform convergence of the continued fraction (4.1), with the exception of a finite number of points of inessential divergence, an any domain $TC\overline{T}$, where $\overline{T}$ is the essential divergence, in any domain $T \subset \overline{T}$, where $\overline{T}$ is the complex plane cut along that segment of the line $\arg(z) = \arg(c)$ which joins the point $-(4c)^{-1}$ and the point at infinity and which does not pass through the Origin.

CHAPTER V

CONTINUED FRACTION SOLUTION OF DIFFERENTIAL EQUATIONS

The equation in its Standard form is taken as

$$Y'' + P(x)Y' + Q(x)Y = 0$$

Solve it for Y and get Y =

$$F_0(x)y' + G_1(x)y''; \quad F_0 = \frac{-P}{Q}; \; G_1 = \frac{-1}{Q}.$$

One differentiation gives y' =

$$= F_1(x)y'' + G_2(x)y''', \text{ where } F_1(x) =$$

$$+ \left(F_0 + G_1'\right) \Big/ \left(1 - F_0'\right); \; G_2(x) = G_1 \Big/ \left(1 - F_0'\right)$$

After n differentiations, $y^n = f_n y^{n+1} + G_{n+1} y^{(n+2)}$,

where $F_n = \left(F_{n-1} + G_n'\right) \Big/ \left(1 - F_{n-1}'\right)$, $G_{n+1} = G_n \Big/ \left(1 - F_{n-1}'\right)$,

From these equations, it follows that

$$\frac{Y}{Y'} = F_0 + G_1 \frac{y''}{y'} = F_0 + G_1 \Big/ \left(F_1 + \frac{G_2 y'''}{y''}\right) \cdots$$

$$= F_0 + \frac{G_1}{F_1} + \frac{G_2}{F_2} + \frac{G_n}{(F_n + R_n)}$$

Invert the equation for $\frac{y}{y^1}$ and get

$$\frac{y^1}{y} = \frac{d(\ln y)}{dx} = \frac{1}{F_0} + \frac{G_1}{F_1} + \frac{G_2}{F_2} + \dots + \frac{Gn}{F_{n+1}} + \dots$$

a. If n is finite, one integration of the last equation gives lny; hence the solution of the differential equation.

b. The continued fraction converges to $\frac{y^1}{y}$ if the following conditions are met;

i . $y \neq 0$

ii F_m, G_m approach limits F, G as n approaches infinity.

iii The roots S_1 S_2 of $S^2 - F S + G = 0$ are of unequal moduli.

iv $|S_2| < |S_1|$ ∴ limit $|y^{(n)}|^{\frac{1}{n}} < |S_2|^{-1}$, $|S_2| \neq 0$.

If the limit is finite, $|S_2| = 0$.

EXAMPLE

$$(2x - x^2)y'' + (x^2 - 2)\, y' + 2(1-x)y = 0$$

$$y'' + \frac{x^2 - 2}{2x - x^2}\, y' + \frac{2(1 - x\,)}{2x - x^2}\; y = 0$$

Solving for y

$$y = \frac{\dfrac{-x^2 + 2}{2x - x^2}}{\dfrac{2(\,1-x)}{2x - x^2}}\, y' \qquad + \qquad \frac{\dfrac{1}{2(\,1-x)}}{2x - x^2}\, y''$$

$$\frac{2x - x^2}{2(1-x\,)}\; y''$$

$$F_0 = \frac{-P}{Q} = \frac{\dfrac{x^2 - 2}{2x - x^2}}{\dfrac{2(1-x)}{2x - x^2}}\, y' \qquad G_1 = \frac{-1}{Q} = \frac{\dfrac{-1}{2x - x^2}}{\cancel{2(1-x^2)}}\, y''$$

$$Y = \left(\frac{x^2 + 2}{2(1-x)}\right) y' \qquad + \qquad \left[\frac{-(2x - x^2)}{2(1-x)}\right] y''$$

one differential gives

$$y' = \frac{(x^3 - x^2 - 2x + 2)y'' + (x^2 - 2x + 2)y' + (-2x + 3x^2 - x^3)y''' + (-2 + 2x - x^2)y''}{\left[2(1-x)\right]^2}$$

$$\left[1 - \frac{(x^2 - 2x + 2)}{\left[2(1-x)\right]^2}\right] y' = \frac{(-2x + 3x^2 - x^3)y''' + (x^3 - 2x^2)y''}{\left[2(1-x)\right]^2}$$

$$\left\{\frac{x^2 - 2x}{\left[2(1-x)\right]^2}\right\} y' = \frac{x^3 - 2x^2}{\left[2(1-x)^2\right]}\; y'' + \frac{(-x^3 + 3x^2 - 2x)}{\left[2(1-x)\right]^2}\; y'''$$

where

$$F_1(x) = \frac{\dfrac{-x^2+2}{2(1-x)} + \left(\dfrac{-2x^2+x^2}{2(1-x)}\right)'}{1 - \left(\dfrac{-x^2+2}{2(1-x)}\right)'}$$

$$F_1(x) = \frac{x^3}{-2x^2+x^2}$$

$$G_2(x) = \frac{\dfrac{-2x+x^2)}{2(1-x)}}{1-\left(\dfrac{x^2+2}{2(1-x)}\right)'}$$

$$G_2(x) = 1-x$$

$$F_2(x) = \frac{\dfrac{x^3}{-2x+x^2} + 1-x}{1 - \left(\dfrac{x^3}{-2x+x^2}\right)'}$$

$$= \frac{\dfrac{x^3}{-2x+x^2} + (-1)}{1 - \left(\dfrac{7x^4-6x^3}{-2x+x^2 \quad 2}\right)}$$

$$F_2(x) = \frac{x^3-3x^2+4x-4}{2(-3x^2+x+2)}$$

$$\frac{y}{y'} = \frac{-x^2+2}{2(1-x)} + \frac{\left[\dfrac{-2x+x^2}{2(1-x)}\right]y''}{y'} = \frac{\dfrac{-x^2+2}{2(1-x)} + \dfrac{2x+x^2}{2(1-x)}}{\dfrac{x^3-2x^2}{\left[2(1-x)\right]^2} + \dfrac{(-x^3+3x^2-2x)y'''}{y''}}$$

$$= \frac{-2x+2}{\dfrac{x^3-2x^2}{\left[2(1-x)\right]^2} + \dfrac{(-x^2+3x^2-2x)y'''}{y''}}$$

$$= \frac{-x^2 + 2}{2(1-x)} + \cfrac{\frac{-2x + x^2}{2(1-x)}}{\cfrac{x^3}{-2x + x^2} + \cfrac{1 - x}{\cfrac{x^3 - 3x^2 + 4x - 4}{2(-3x^2 + x + 2)}}} + \cdots$$

$$= \frac{-x^2 + 2}{2(1-x)} + \frac{4 - 4x + x^2}{2x\ (1-x)} + \frac{6x^3 - 8x^2 - 2x + 4}{x^3 - 3x^2 + 4x - 4} + \cdots$$

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